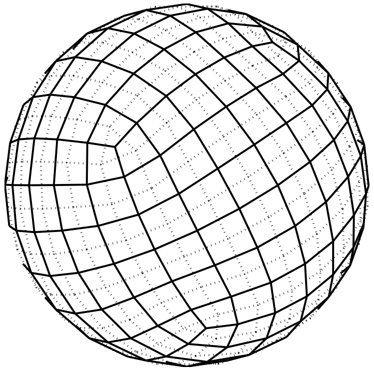
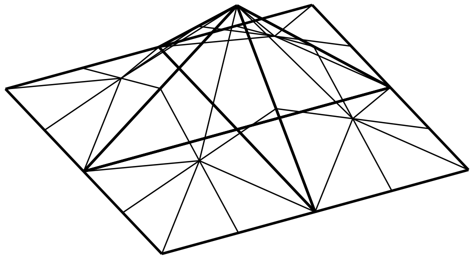
Modelling the atmosphere with compound compatible finite elements





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Applied Geometric Mechanics network
Geometric and Structure-Preserving Numerics
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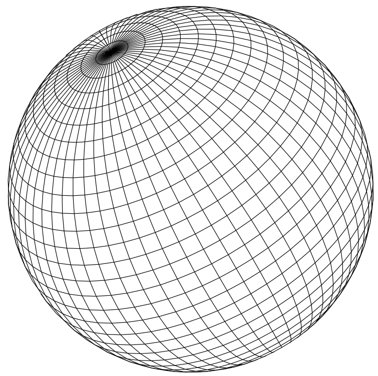
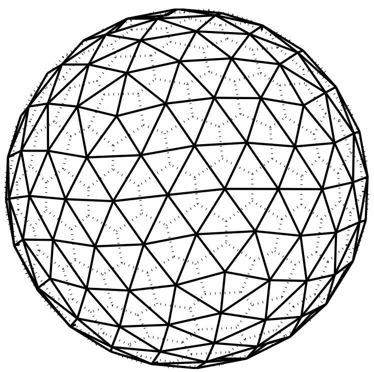
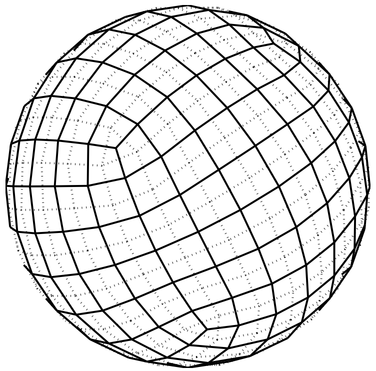
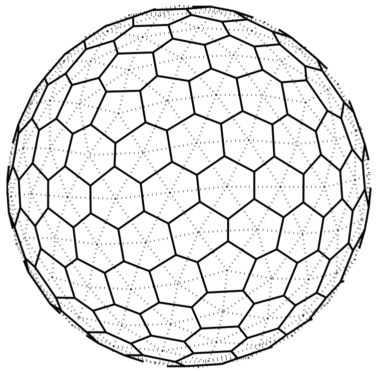
We need a quasi-uniform spherical mesh

Resolution clustering at the poles of the lat-long grid will lead to a communication bottleneck on future supercomputers...









...but schemes that work well on a lat-long grid exploit orthogonality of the coordinate system and quadrilateral structure.

Shallow water equations

$$\begin{aligned}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{f} &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{q}^{\perp} + \nabla \left(|\mathbf{u}|^2/2 + \phi_T \right) &= 0\end{aligned}$$

where

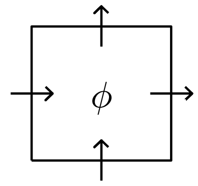
$\mathbf{f} = \mathbf{u}\phi$, mass flux

$\mathbf{f}^{\perp} = \mathbf{k} \times \mathbf{f}$, where $\mathbf{k}$ is the unit vertical vector

$\xi = \mathbf{k} \cdot \nabla \times \mathbf{u}$, relative vorticity

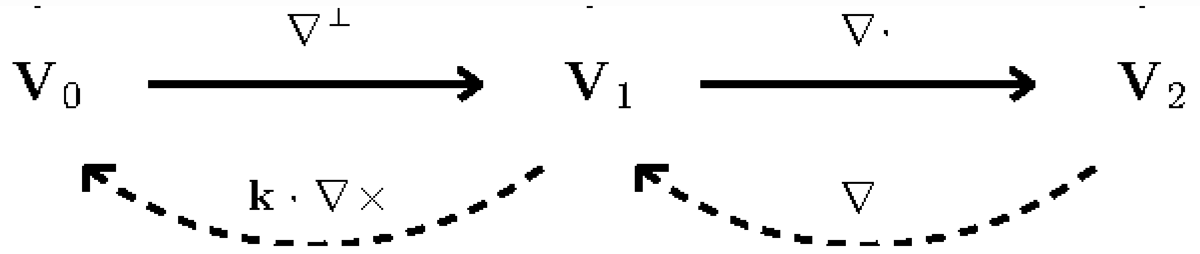
$\pi = (f + \xi)/\phi$, potential vorticity: $\frac{\partial}{\partial t}(\phi\pi) + \nabla \cdot \mathbf{q} = 0$ and $\mathbf{q} = \mathbf{f}\pi$

Relating physical to mathematical/numerical properties

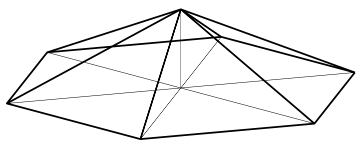
Wave propagation	C-grid 
Mass conservation	Flux form mass equation
(Linear) energy conservation	$\nabla\cdot$ and $-\nabla$ adjoints, $\int \mathbf{f}\cdot\mathbf{f}^{\perp}\,\mathrm{d}A = 0$, ...
Steady geostrophic modes	$\nabla\phi$ spans irrotational velocity space; vorticity equation sees 'correct' divergence
$\nabla(\phi + KE)$ not a ξ source	$\nabla\times\nabla = 0$
PV advection property	(only for SW)

(There are additional requirements in 3D.)

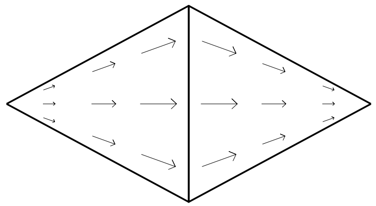
de Rham complex



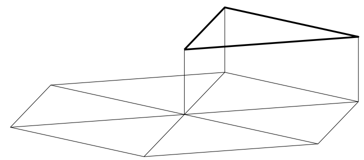
Choose finite elements that live in three function spaces with special properties.



ξ



$\mathbf{u}$



ϕ

∇ and $\mathbf{k}\cdot\nabla\times$ are defined by integration by parts. E.g.,

$$\mathbf{g} = \nabla\phi, \qquad \phi \in \mathrm{V}_2, \quad \mathbf{g} \in \mathrm{V}_1$$

means

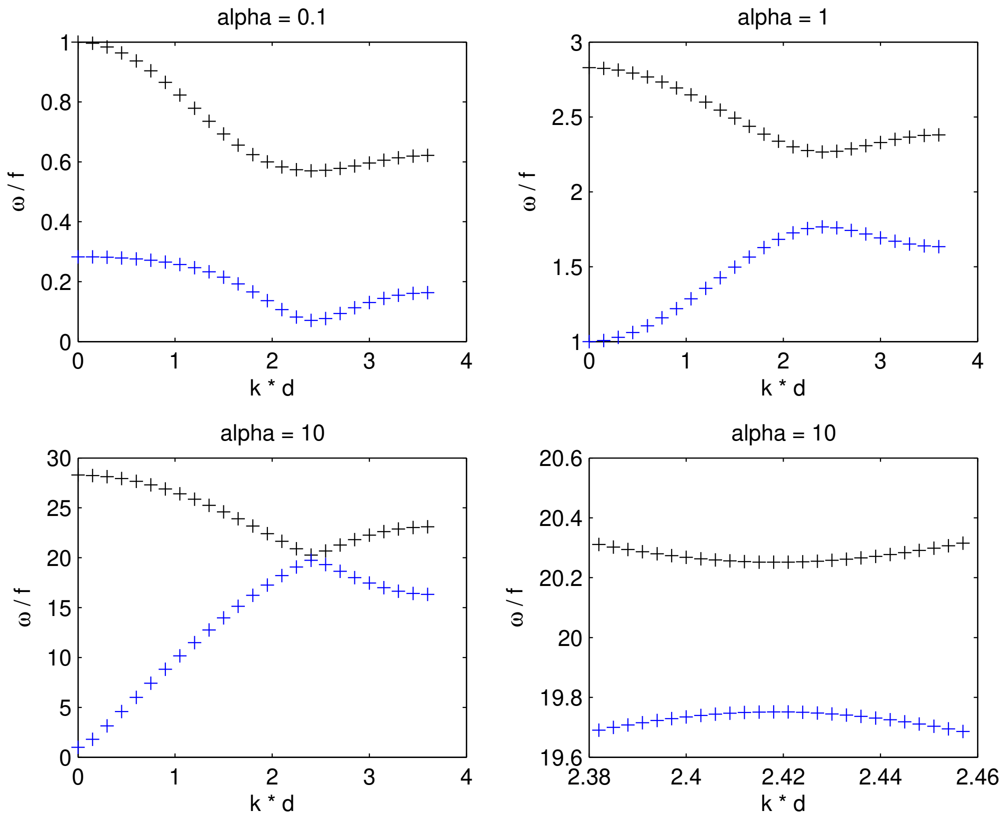
$$\int \mathbf{g}\cdot\mathbf{v}\,\mathrm{d}A = \int \mathbf{v}\cdot\nabla\phi\,\mathrm{d}A = -\int \phi\nabla\cdot\mathbf{v}\,\mathrm{d}A \qquad \forall\,\mathbf{v} \in \mathrm{V}_1$$

The problem with the triangular C-grid ...

The number of geopotential plus divergence DoFs (= number of cells) is greater than twice the number vorticity DoFs (= number of vertices).

The numerical dispersion relation has extra gravity wave branches, and these are badly behaved.

Here α is Rossby radius divided by distance between triangle centres.



The numerical dispersion relations for quadrilateral and hexagonal C-grids are better behaved.

However, ...

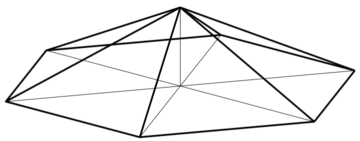
- Neither planar quadrilaterals nor planar hexagons wrap the sphere nicely.

Quadrilaterals can be **bent** so as to wrap the sphere nicely. However, in practice this requires the use of Piola transforms to map between planar reference elements and bent physical elements. Consequently, the basis elements in V_2 are no longer piecewise constant, and the scheme is no longer consistent unless **'rehabilitation'** is applied.

- As we'll see, we'd like to use a **dual mesh** too, and we'll need to be able to compute inner products between the original (**primal**) and dual mesh elements.

This motivates the idea to build **compound** quadrilateral or hexagonal elements out of planar triangular elements.

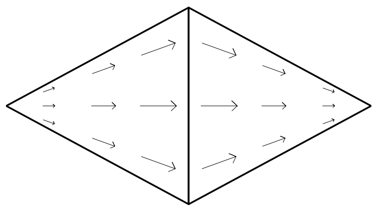
Harmonic extension (Christiansen 2008)



Define the basis function γ_j to equal 1 at vertex j and 0 at all other vertices.

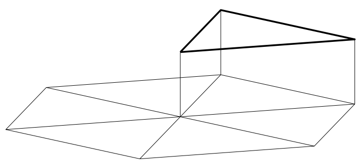
Extend harmonically along edges, i.e., $\mathrm{d}^2\gamma_j/\mathrm{d}s^2 = 0$.

Extend harmonically into the element interior: $\nabla^2\gamma_j = 0$.



Define the basis function $\mathbf{v}_e$ to have normal component 1 along edge e and 0 at all other edges.

Extend harmonically into the interior of the element, i.e., $\nabla(\nabla \cdot \mathbf{v}_e) = \mathbf{0}$ and $\mathbf{k} \cdot \nabla \times \mathbf{v}_e = 0$.

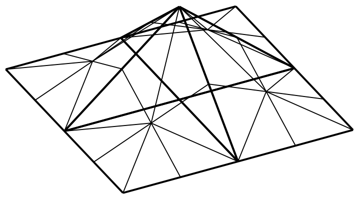
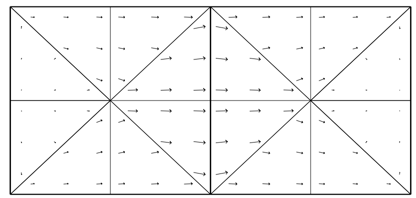
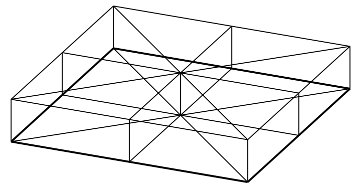
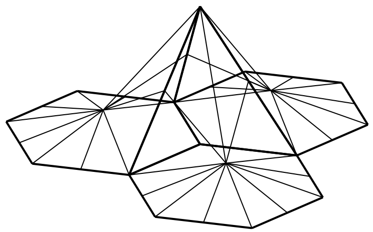
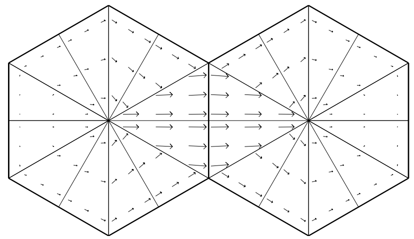
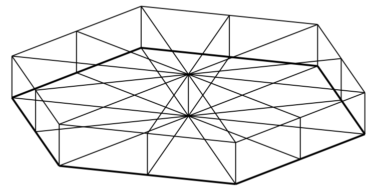
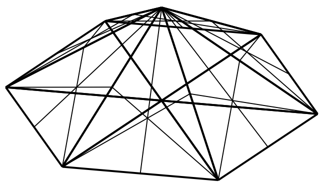
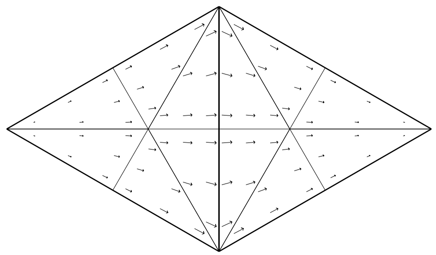
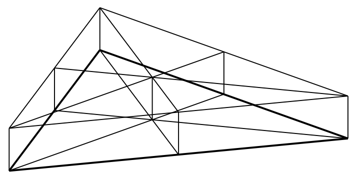


Define the basis function α_i to equal 1 in cell i.

• Harmonic extension clearly generates the triangular compatible finite elements we met earlier.

• It also works (i.e., generates a de Rham complex) on arbitrary polygons. Unfortunately, the basis functions do not have analytical expressions.

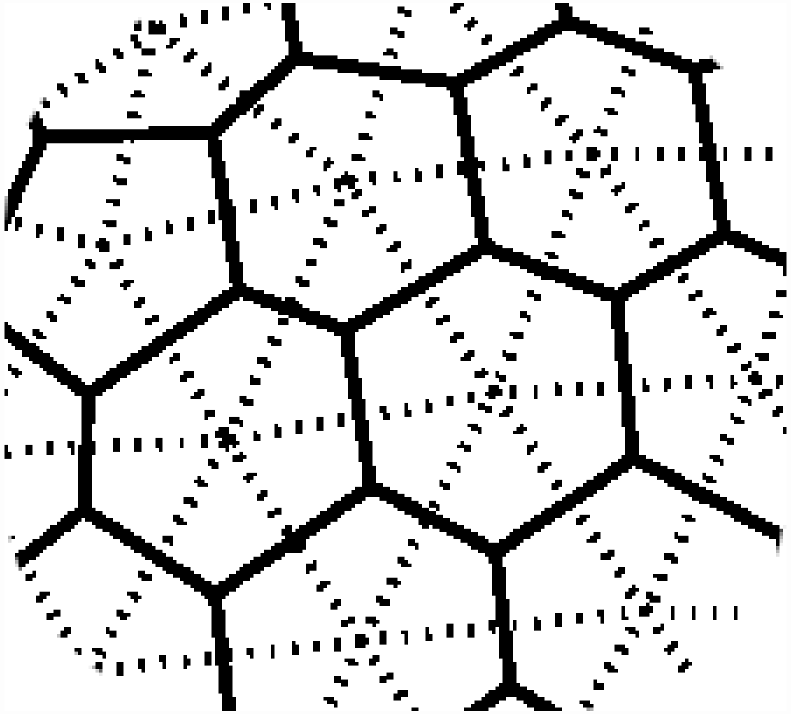
• But **discrete harmonic extension** also generates a de Rham complex, i.e., solve $\nabla^2 \gamma_j = 0$ etc. numerically using compatible finite elements on a **supermesh** of triangular elements.

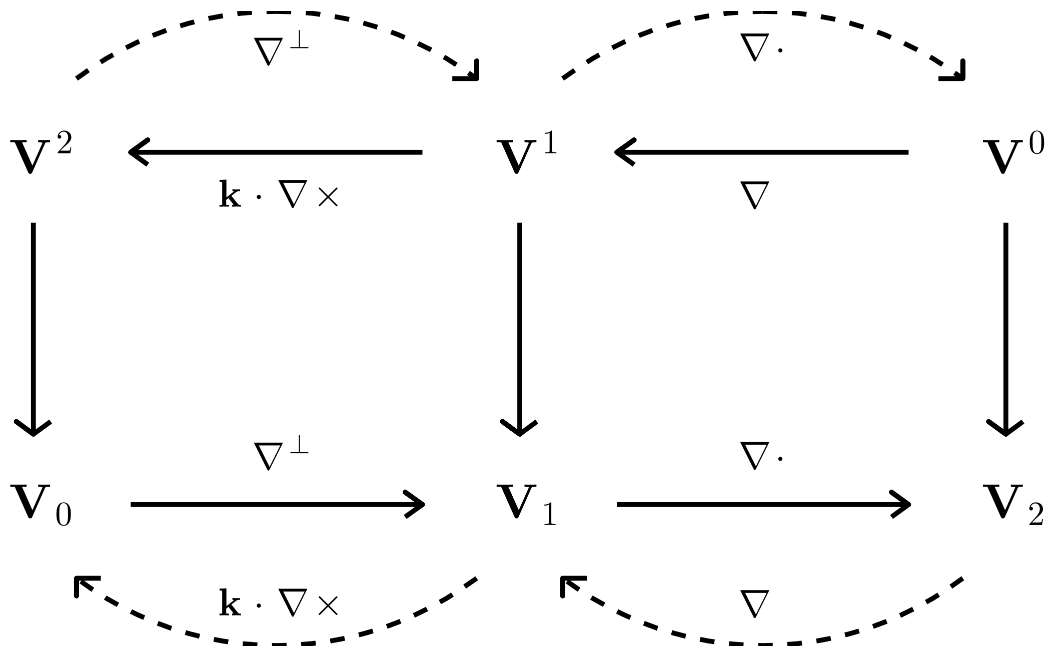
Dual meshes

We'd like to use a finite volume scheme for advection of ϕ. This is straightforward because $\phi \in \mathrm{V}_2$ is piecewise constant, while for $\mathbf{u} \in \mathrm{V}_1$ we have the normal velocity components.

We'd also like to use a finite volume scheme to compute the PV flux. This is tricky because vorticity naturally lives in V_0, while for the mass flux $\mathbf{f} \in \mathrm{V}_1$ we have the components tangential to the dual cells



We can work with **two families** of compatible finite elements, one on the primal mesh and one on the dual mesh.



Matrix notation

Let $\phi = \sum_i \phi_i \alpha_i \in \mathrm{V}_2$ and write Φ for the vector of DoFs.

Let $\tilde{\phi} \in \mathrm{V}_0$ and write $\tilde{\Phi}$ for the vector of DoFs.

Let $\bar{\phi} \in \mathrm{V}^2$ and write $\bar{\Phi}$ for the vector of DoFs.

Let $\mathbf{u} \in \mathrm{V}_1$ and write U for the vector of DoFs.

Let $\hat{\mathbf{u}} \in \mathrm{V}^1$ and write $\hat{U}$ for the vector of DoFs.

Let $U^{\perp}$ be the vector of DoFs of $\mathbf{u}^{\perp}$ projected into V_1

Strong derivatives

$$\delta = \nabla \cdot \mathbf{u} \quad \Rightarrow \quad \Delta = D_2 U$$

$$\mathrm{u} = \nabla^{\perp}\psi \quad \Rightarrow \quad U = -D_1 \Psi$$

$$\hat{\mathrm{u}} = \nabla \hat{p} \quad \Rightarrow \quad \hat{U} = \bar{D}_1 \hat{P}$$

$$\hat{\xi} = \mathbf{k} \cdot \nabla \times \hat{\mathbf{u}} \quad \Rightarrow \quad \hat{\Xi} = \bar{D}_2 \hat{U}$$

where $\bar{D}_1 = -D_2^{\mathrm{T}}$ and $\bar{D}_2 = D_1^{\mathrm{T}}$.

The D's are **incidence matrices**, related to the mesh structure.

Averaging operators

$MU^{\perp} = -WU$ where $W_{e\,e'} = \langle \mathbf{v}_e, \mathbf{v}_{e'}^{\perp} \rangle$

$N\tilde{\Phi} = R\Phi$ where $R_{j\,i} = \langle \gamma_j, \alpha_i \rangle$

Hodge stars

$$L\Phi = I\hat{\Phi}$$

$$MU = H\hat{U}$$

$$N\Xi = J\hat{\Xi}$$

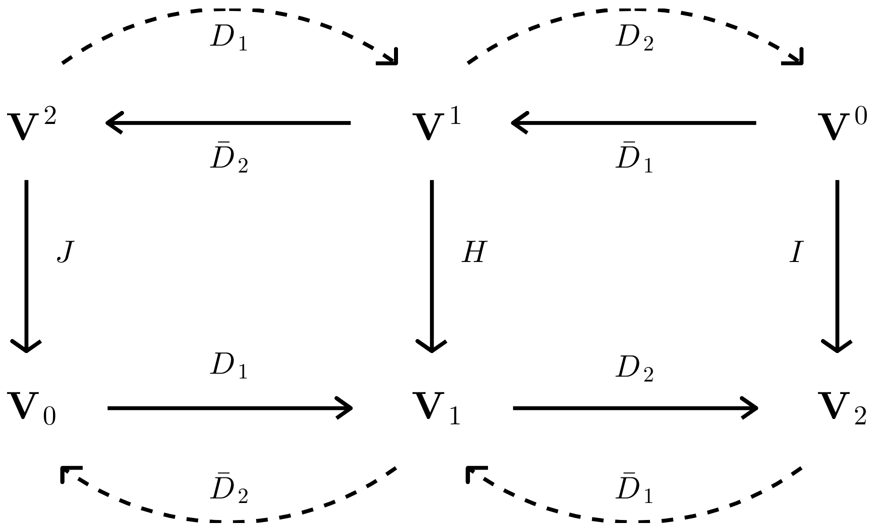
where L, M, and N are mass matrices for V_2, V_1, and V_0, respectively.

Weak derivatives

$$\mathbf{g} = \nabla\phi \quad \Rightarrow \quad MG = \bar{D}_1 L\Phi$$

$$\xi = \mathbf{k}\cdot\nabla\times\mathbf{u} \quad \Rightarrow \quad N\Xi = \bar{D}_2 MU$$

Useful identities



$$\bar{D}_1 = -D_2^{\mathrm{T}} \qquad D_2 D_1 = 0 \qquad \bar{D}_2 H = J \bar{D}_2$$

$$-\bar{D}_2 W = R D_2$$

$$\bar{D}_2 = D_1^{\mathrm{T}} \qquad \bar{D}_2 \bar{D}_1 = 0 \qquad \bar{D}_1 I = H \bar{D}_1$$

Discrete shallow water equations

$$\dot{\Phi} + D_2 F = 0$$

$$M\dot{U} + MQ^{\perp} + \bar{D}_1 L(\Phi + K) = 0$$

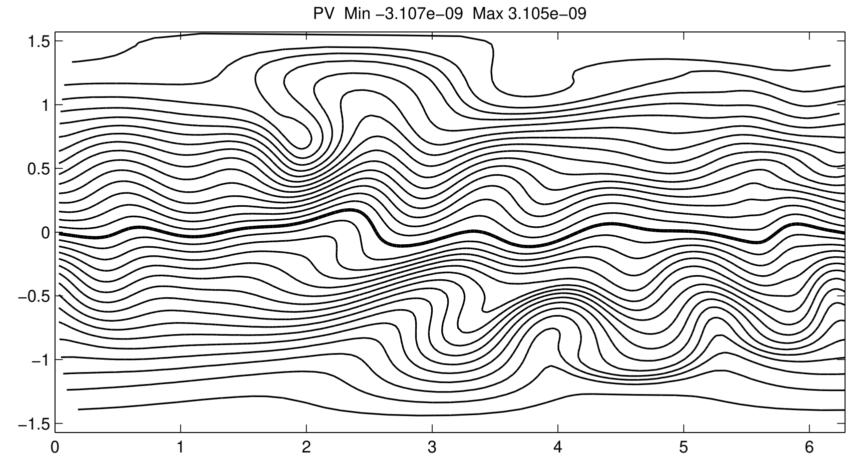
where F is computed using an upwind finite volume scheme on the primal mesh and $\hat{Q}^{\perp}$ is computed using an upwind finite volume scheme on the dual mesh.

A semi-implicit time stepping scheme is used with a quasi-Newton / multi-grid solver.

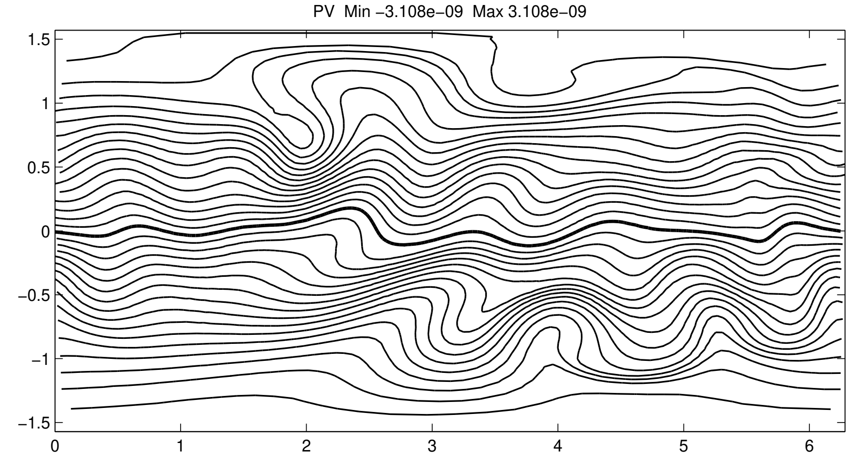
Example results

Flow over an isolated mountain, PV at day 15.

Cube, 13824 cells.



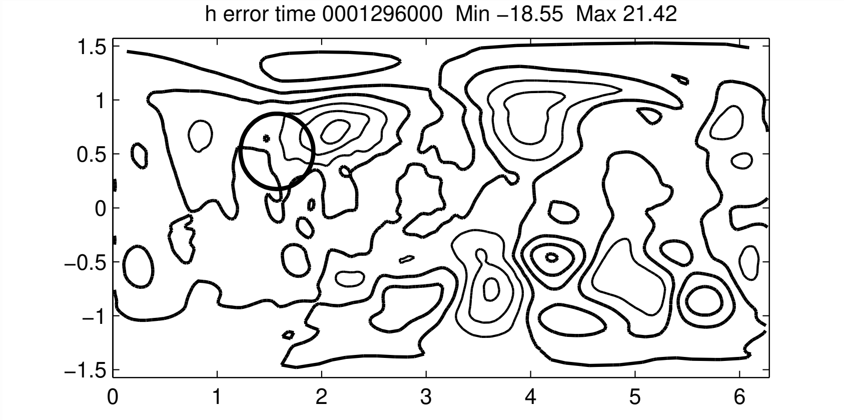
Hex, 10242 cells.



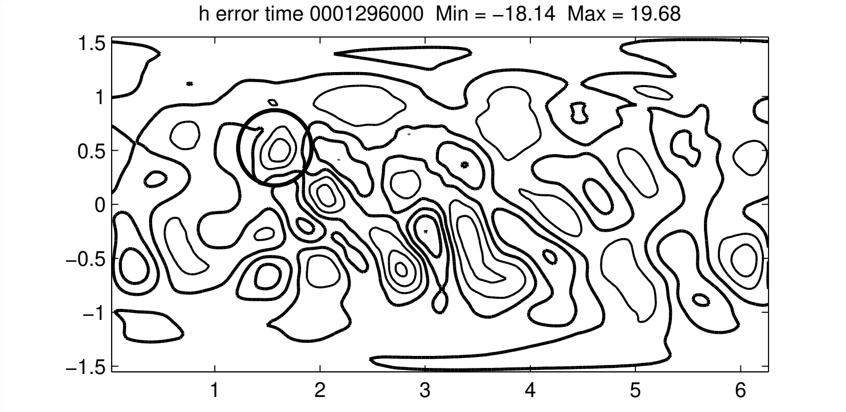
Example results

Flow over an isolated mountain, height error at day 15.

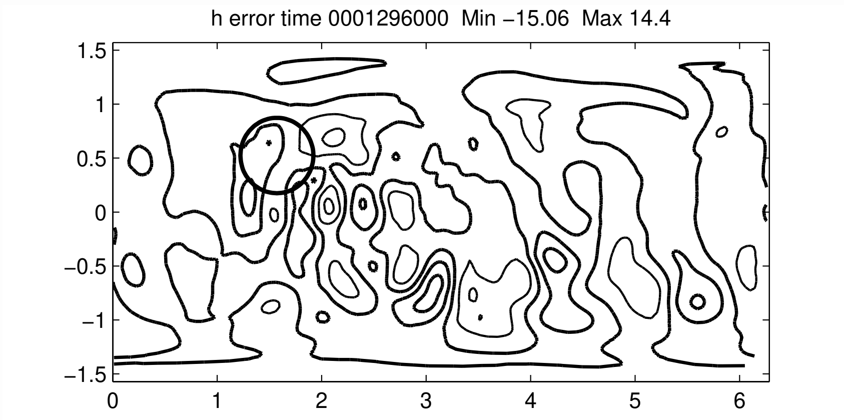
Cube, 13824 cells



SISL lat-long 160x80



Hex, 10242 cells



Results also confirm...

- Mass is conserved to machine precision
- Dual mesh mass and PV behave as they should
- Geostrophic balance is captured accurately; no sign of numerically generated imbalance
- Energy and potential entropy are disspated by the upwind advection scheme in a reasonable way
- On standard test cases accuracy is comparable to a state-of-the-art SISL scheme on a lat-long grid

Limitations

The compound finite elements discussed here are of the lowest order. It appears difficult to construct higher order compound elements with all the desired properties.

In this 2D implementation all matrix operators are precomputed. In 3D it is not obvious that operators can be precomputed, while quadrature on the fly would be expensive.

In 3D it does not appear possible to obtain the PV advection property.